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GEOGRAPHY

Cylindrical Projection Conformality of Triaxial Ellipsoid

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Abstract—It is proved that the projection of a triaxial ellipsoid with rectilinear meridians orthogonal to the rectilinear equator cannot be strictly conformal. The proof is based on the fact that due to the degeneration of the horizontal coordinate in latitude the relation between arc differentials in the projection depends linearly on the relation between the latitude and longitude differentials, but due to the dependence of the vertical coordinate on both latitude and longitude, the direction and certain value of this relation is retained upon the integration path selection. Other directions do not keep the relation between differentials of the corresponding arcs in the triaxial ellipsoid and in the projection plane. A projection keeping the angle between the parallel and the meridian obtained by the integration path selection by the initial meridian and then by the parallel is offered.

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The projection of the triaxial ellipsoid with rectilinear meridians orthogonal to the rectilinear equator developed in [1] and [2] and called "conformal cylindrical" is used in the mapping. This projection in the neighborhood of each meridian is similar to the conformal projection of the rotation ellipsoid corresponding to the meridional section, but according to the investigation data does not keep the angle between the parallel and the meridian.

It is proved in this paper that the projection of the triaxial ellipsoid with rectilinear meridians orthogonal to the rectilinear equator cannot be strictly conformal. A projection keeping the angle between the parallel and the meridian is offered.

Let us assume that **i**, **j**, **k** are unit axes of the coordinate system related to the triaxial ellipsoid center ($\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$ is any vector). The equation of the triaxial ellipsoid surface $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\right)$ in angular planetocentric coordinates (ϕ is latitude and λ is longitude) will take the following form:

$$r = \frac{a}{\sqrt{t}},$$

where

$$t = \cos^2\phi\cos^2\lambda + \frac{\cos^2\phi\sin^2\lambda}{1 - e_1^2} + \frac{\sin^2\phi}{1 - e^2},$$

e_1 is the eccentricity of the equatorial ellipse and e is the eccentricity of the initial meridian section ellipse. The vector **r** derivates are as follows:

$$\frac{\partial\mathbf{r}}{\partial\phi} = \mathbf{i}\frac{\partial x}{\partial\phi} + \mathbf{j}\frac{\partial y}{\partial\phi} + \mathbf{k}\frac{\partial z}{\partial\phi}, \quad \frac{\partial\mathbf{r}}{\partial\lambda} = \mathbf{i}\frac{\partial x}{\partial\lambda} + \mathbf{j}\frac{\partial y}{\partial\lambda} + \mathbf{k}\frac{\partial z}{\partial\lambda}, \quad (1)$$

where $x = r\cos\phi\cos\lambda$, $y = r\cos\phi\sin\lambda$, $z = r\sin\phi$.

The modules and dot products of vectors (1) are as follows:

$$\left|\frac{\partial\mathbf{r}}{\partial\phi}\right| = a\sqrt{\frac{4t^2 + \sin^2 2\phi\left(\cos^2\lambda + \frac{\sin^2\lambda}{1 - e_1^2} - \frac{1}{1 - e^2}\right)^2}{4t^3}} = \sqrt{E},$$

$$\left|\frac{\partial\mathbf{r}}{\partial\lambda}\right| = a\cos\phi\sqrt{\frac{4t^2 + \sin^2 2\lambda\cos^2\phi\left(1 - \frac{1}{1 - e_1^2}\right)^2}{4t^3}} = \sqrt{G}, \quad (2)$$

$$\frac{\partial\mathbf{r}}{\partial\phi}\cdot\frac{d\mathbf{r}}{\partial\lambda} = \frac{a^2\sin 2\phi\sin 2\lambda\cos^2\phi\left(\cos^2\lambda + \frac{\sin^2\lambda}{1 - e_1^2} - \frac{1}{1 - e^2}\right)\left(1 - \frac{1}{1 - e_1^2}\right)}{4t^3} = F,$$

where E, G, F are Gauss ratios of the first quadratic form.

The angle ω sine and cosine between the parallel and the meridian are calculated in the following way:

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$$\cos\omega = \frac{F}{\sqrt{EG}}, \quad \sin\omega = \sqrt{1 - \frac{F^2}{EG}}.$$

Sine is always positive.

Let us calculated the angle γ between the meridian and an arbitrary curve in the triaxial ellipsoid, i.e., between the vectors $\frac{\partial \mathbf{r}}{\partial \phi}$ and $d\mathbf{r} = \mathbf{i}dx + \mathbf{j}dy + \mathbf{k}dz$ tangent to this curve at the point with coordinates ϕ, λ, where

$$dx = \frac{\partial x}{\partial \phi}\frac{d\phi}{d\lambda}d\lambda + \frac{\partial x}{\partial \lambda}d\lambda, \quad dy = \frac{\partial y}{\partial \phi}\frac{d\phi}{d\lambda}d\lambda + \frac{\partial y}{\partial \lambda}d\lambda,$$

$$dz = \frac{\partial z}{\partial \phi}\frac{d\phi}{d\lambda}d\lambda + \frac{\partial z}{\partial \lambda}d\lambda.$$

Taking into account formulas (2), we will obtain

$$\cos\gamma = \frac{\frac{\partial \mathbf{r}}{\partial \phi}\cdot d\mathbf{r}}{\left|\frac{\partial \mathbf{r}}{\partial \phi}\right||d\mathbf{r}|} = \frac{\left(\frac{\sqrt{E}}{\sqrt{G}}\frac{d\phi}{d\lambda} + \frac{F}{\sqrt{EG}}\right)}{\sqrt{\frac{E}{G}\left(\frac{d\phi}{d\lambda}\right)^2 + 1 + \frac{2F}{G}\frac{d\phi}{d\lambda}}}.$$

Hence, the derivative along the γ direction is as follows:

$$\frac{d\phi}{d\lambda} = (-\cos\omega + \cot\gamma\sin\omega)\frac{\sqrt{G}}{\sqrt{E}}. \tag{3}$$

In the case with the arbitrary curve, the angle γ changes along it. If the angle is constant, then relation (3) is a differential loxodrome equation.

The plane (X_{plane}, Y_{plane}), containing the vectors (1) and point with ellipsoid surface coordinates ϕ, λ, is tangent to the ellipsoid at this point. Let us select the coordinate system in which the Y_{plane} axis is directed along the vector $\frac{\partial \mathbf{r}}{\partial \phi}$ (along the meridian) and the X_{plane} axis is directed rightwards transversely to the meridian direction. Then

$$dx_{plane} = \frac{dx_{plane}}{d\lambda}d\lambda = \left|\frac{\partial \mathbf{r}}{\partial \lambda}\right|\sin\omega d\lambda = \sqrt{G}\sin\omega d\lambda,$$

$$dy_{plane} = \frac{\partial y_{plane}}{\partial \phi}d\phi + \frac{\partial y_{plane}}{\partial \lambda}d\lambda$$

$$= \left|\frac{\partial \mathbf{r}}{\partial \phi}\right|d\phi + \left|\frac{\partial \mathbf{r}}{\partial \lambda}\right|\cos\omega d\lambda = \sqrt{E}d\phi + \sqrt{G}\cos\omega d\lambda, \tag{4}$$

$$\frac{dy_{plane}}{dx_{plane}} = \frac{\sqrt{E}d\phi + \sqrt{G}\cos\omega d\lambda}{\sqrt{G}\sin\omega d\lambda} = \cot\omega + \frac{\sqrt{E}}{\sqrt{G}\sin\omega}\frac{d\phi}{d\lambda}.$$

Inserting the relation between $d\phi$ and $d\lambda$ from (3) corresponding to the γ direction into equation (4), we obtain $\frac{dy_{plane}}{dx_{plane}} = \cot\gamma$, which demonstrates the retention of angles under the transition to the tangent plane.

In the projection plane, the coordinate axis X_{proj} is directed horizontally rightwards, while Y_{proj} is directed vertically upwards. The horizontal coordinate x_{proj} depends only on λ; hence:

$$dx_{proj} = \frac{dx_{proj}}{d\lambda}d\lambda, \quad x_{proj}(\lambda_i) = \int_0^{\lambda_i} dx_{proj},$$

where $\frac{dx_{proj}}{d\lambda}$ is a function in λ. When passing to the (X_{proj}, Y_{proj}) plane, it is necessary to fulfill the condition of $\frac{dy_{proj}}{dx_{proj}} = \frac{dy_{plane}}{dx_{plane}}$ to observe the conformality of normal cylindrical projection, in other words

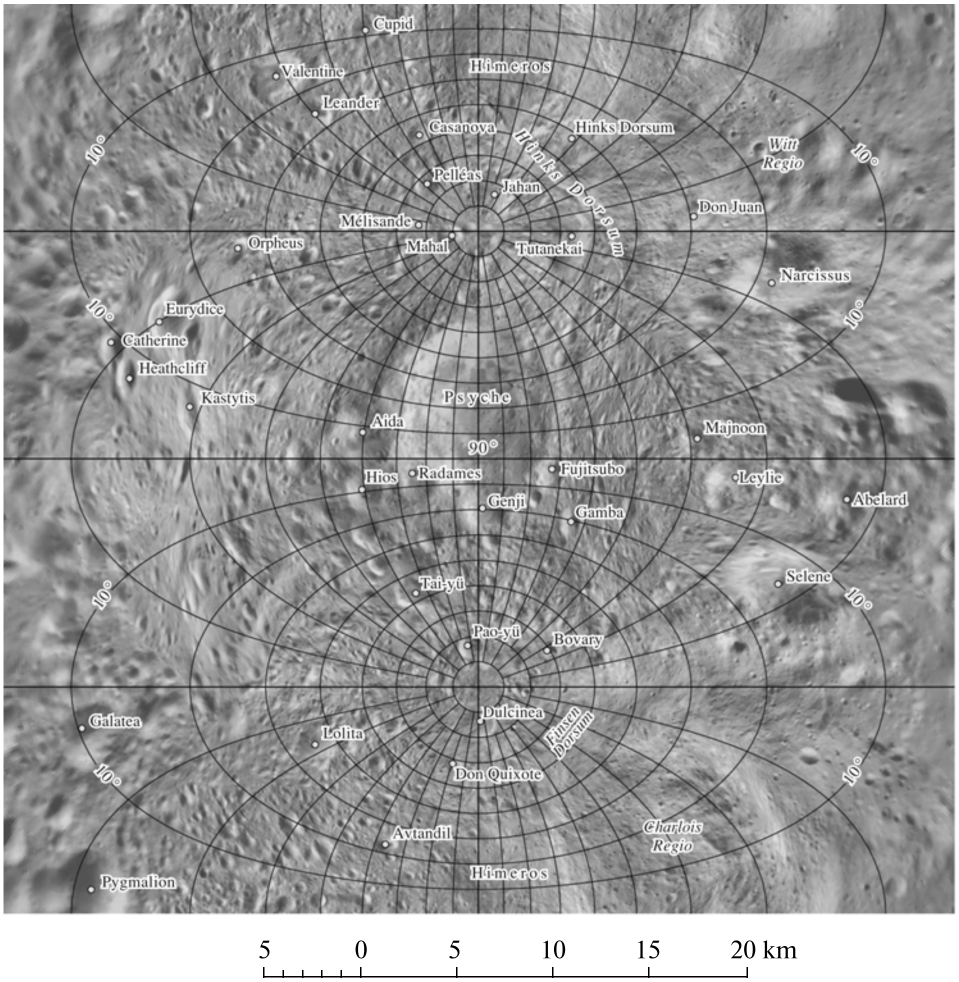
$$dy_{proj} = \frac{dx_{proj}}{d\lambda}\frac{\sqrt{E}}{\sqrt{G}\sin\omega}d\phi + \frac{dx_{proj}}{d\lambda}\cot\omega d\lambda. \tag{5}$$

It is necessary to select the integration path to obtain the vertical coordinate y_{proj} due to the dependence of dy_{proj} on two variables $d\phi$ and $d\lambda$. It should be noted that the degeneration of the cylindrical projection coordinate x_{proj} along one of the ellipsoid coordinates (ϕ) results in the linear dependence of $\frac{dy_{proj}}{dx_{proj}}$ on $\frac{d\phi}{d\lambda}$. Meanwhile, upon the integration path selection, the direction, i.e., the certain $\frac{d\phi}{d\lambda}$ value, is retained at each point. In this case, the angle in the projection plane between the meridian and the selected direction is equal to the corresponding angle in the tangent plane. Meanwhile, the equality of relations between differentials on the tangent plane and in the projection will not be valid for any other $\frac{d\phi}{d\lambda}$ value. Hence, it is impossible to construct the conformal cylindrical projection of the triaxial ellipsoid, whose meridians are straight lines perpendicular to the rectilinear equator.

When selecting the integration path by the equator and then by the meridian, we use only the first member from the right side of Equation (5) to obtain the vertical coordinate as in [2]. In this case, the conformality of angles between the meridian and the parallel in the projection plane and the ellipsoid is not retained. Under the integration by the initial meridian and then by the parallel, the angle between the meridian and the parallel in the projection plane is equal to the corresponding angle in the ellipsoid. The formulas of cylindrical projection keeping the angle between the meridian and the parallel and also lengths in the equator take the following form (implemented in [3]):

$$x_{proj}(\phi_i, \lambda_i) = \int_0^{\lambda_i} \sqrt{G_0}d\lambda,$$

$$y_{proj}(\phi_i, \lambda_i) = y_0(\phi_i, \lambda = 0) + \int_0^{\lambda_i} \cot\omega(\phi_i, \lambda)\sqrt{G_0}d\lambda,$$



433 Eros asteroid surface.

where $y_0(\phi_i, \lambda = 0) = \int_0^{\phi_i} \frac{a\sqrt{E}}{\sqrt{G}} d\phi$ and G_0 is the Gaussian ratio G in the equator.

Under the transversal projection orientation, the major equatorial axis becomes polar, and lengths are retained at meridians of 90° and 270°. If the minor equatorial axis of the initial ellipsoid is equal to its polar axis, then the obtained projection is, in fact, the rotation ellipsoid projection. The figure demonstrates the 433 Eros asteroid surface in the transverse conformal cylindrical projection of the triaxial ellipsoid with account for the fact that the longitude of this celestial body is counted westwards.

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